The Problem Cauldron

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Preface

This is a collection of some of the problems I've made. The Main problems usually are the good ones, while the Variations are also high quality. Extensions and Generalizations are for problems which are specific cases, though the generalizations usually hint towards the intended method. Rough drafts are either low quality, obvious, trivial, or annoying to do, and starting points are basically hints that show you some steps, so feel free to ignore those. (Those are just here to illustrate the process I used to come up with problems and to provide an outline for the solutions, since I'll probably take forever to get the solutions out.)

Algebra

1 Find a polynomial with integer coefficients with root $3 + \sqrt[4]{2}$.

2 Find a polynomial with integer coefficients with root $\sqrt{3} + \sqrt[4]{2}$.

3 (Main) Find the smallest natural number n such that $(\sqrt{6}+\sqrt{2}+i\sqrt{6}-i\sqrt{2})^n$ is strictly real.

3.1 (Variation) The smallest positive integer value that $(\sqrt{6}+\sqrt{2}+i\sqrt{6}-i\sqrt{2})^n$, where n is a natural number, can take is x. Find the remainder of x when divided by 1000. (dchen Mock AIME SL)

4 Consider sets $A = \{a_1, a_2...a_n\}$ and $B = \{b_1, b_2...b_n\}$, such that all a and b terms can independently be either -1 or 1. How many distinct pairs of sets A and B exist such that $\{a_1, a_2...a_n\}$ can be mapped in any order to $\{x_1, x_2...x_n\}$ and that $\{b_1, b_2...b_n\}$ can be mapped in any order to $\{y_1, y_2...y_n\}$ such that $(x_1y_1 + x_2y_2 + ... + x_ny_n)^2 = n(y_1^2 + y_2^2 + ... + y_n^2)$? (dchen Algebraic Inequalities)

5 Find the minimum value of $3 + 3^x + 4 + 4^x + 2 + 2^{-x} + 6 + 6^{-x}$. (dchen Algebraic Inequalities)

6 Dan and Tom are playing a coin-flipping game, and Dan flips first. The first person to flip a heads is the winner. Dan's probability of flipping heads is $\frac{1}{3}$ and Tom's chance is n. If Dan and Tom have an equal probability of winning, what is n?

7 Find $\sum_{n=1}^{2017} \frac{-1^n}{n}$, to the nearest integer.

8 Positive reals $\{a_1, a_2, \cdots a_{100}\}$ multiply out to $\frac{13^{50}}{7^{50}}$ Find the smallest possible value of $a_1 + a_2 + \cdots + a_{100}$. (Algebraic Inequalities)

9 Prove that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{99}}{a_{100}} + \frac{a_{100}}{a_1} \geq 100$ given that $a_1, a_2 \cdots a_{100} > 0$. (Algebraic Inequalities)

10 Prove that $-\sqrt{2} \cdot \sin(x) \geq \cos^2(x) - \frac{3}{2}$.

Counting

1 Consider set $\{1, 2, 3..., 12, 13\}$. It is possible to create S distinct sums by adding together N distinct numbers. Find the sum of all values of N that maximize the value of S.

2 A *tweenie* is a natural number that is the mean of two distinct powers of two. Find the tenth smallest tweenie.

3 Consider a regular hexagon with side length 10. How many ways can we put congruent equilateral triangles in the hexagon, if the following two conditions must be satisfied?
a) Each triangle must have a side length greater than 1.
b) The hexagon must be completely covered without any triangles overlapping, or going outside of the hexagon.

4 There are 3 six-sided dice, one red, white, and blue. They are considered distinct. How many ways can the sum of the 15 faces showing on the three die equal 56, if each die orientation is only considered unique if the sum of its faces are unique?

5 The Angry Tomatoes are practicing by splitting their 6 player team into 2 teams of 3 players each and letting their 3 player teams play against each other. The players have skill levels of $1, 2, 3, 4, 5$, and 6, with 1 being the least skillful and 6 being the most skillful. Team captains Tony and Rosa take turns choosing players for their team with Tony choosing first. However, the captains are humans and they can make mistakes. Each time a captain chooses a player, there is a $\frac{1}{3}$ probability that the captain underestimates a player's skill level by 1 and a $\frac{1}{3}$ probability that the captain overestimates a player's skill level by 1. The captains will always choose the player that they perceive as most skillful. If

there is a tie, the captains will choose the player that is actually more skillful. If the team with a higher total skill level wins, what is the probability that Rosa's team wins?

6 Have $p(n)$ be the probability that after rolling a regular 6 sided die n times, you get at least one 6. Find $p(1) + p(2) + ... + p(10)$, to the nearest integer.

Geometry

1 (Main) Consider $\triangle ABC$ with D on BC. Let M, N be the circumcenters of $\triangle ABD, \triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

1.1 (Variation) Consider points A, B, C, D with B, C, D collinear. Let M, N be the circumcenters of $\triangle ADB, \triangle ACD$, respectively. Let the C altitude of $\triangle ABC$ (that is, the line through C perpendicular to AB) intersect AM at H. Prove H, M, D, N are concyclic (that is, there exists a circle that all of these points lie on.)

1.2 (Rough Draft 1) If $\angle A > \angle B$, prove that $\angle MHD = \angle ACB$.

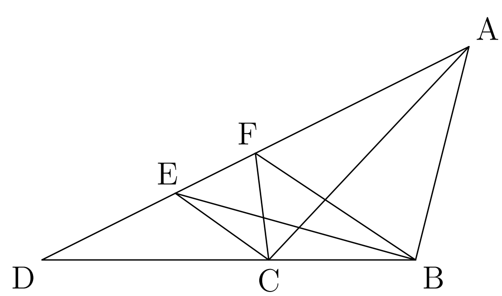
1.3 (Rough Draft 2) If $\angle A < \angle B$, prove that $\angle MHD + \angle ACB = 180°$.

1.4 (Starting Point 1) Prove $\triangle ABC \sim \triangle AMN$.

1.5 (Starting Point 2) Prove $\triangle DMN \cong \triangle AMN$.

1.6 (Triangle Facts) Prove that the ratio of similarity of $\triangle AMN : \triangle ABC$ is minimized when AD is an altitude.

2 In the following diagram, $m\angle BAC = m\angle BFC = 40°$, $m\angle ABF = 80°$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$? (Mock Memorial Day AMC 10)



3 Given points A, B, C, D, E such that BE is the angle bisector of ABC, $\angle AEB =$

$\angle CEB$, $\angle BAC + \angle BDC = \angle ABD + \angle ACD$, and $\angle ADC = 48°$, find $\angle BCA$.

4 Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

5. Consider $\odot M$ and $\odot N$ with no intersections, and with $\odot M$ having a smaller area than $\odot N$. Let $\odot M$ have radius m and $\odot N$ have radius n. Let the locus of points X such that $\frac{j}{MX} = \frac{k}{NX}$ intersect MN at Y, Z, such that $YN < ZN$. Then choose A, B such that the angle bisectors of $\angle YAZ, \angle YBZ$ intersect at M. Then let the circle with diameter MY intersect $\odot M$ at H, I. Have AZ intersect BY at C, have BZ intersect AY at D, and have $\odot N$ intersect AY, BY at E, F. Prove that $AH = BI = CF = DE$.

6. Consider four concyclic points A, B, C, D. Then let AD, BD, CD intersect BC, CA, AB, respectively, at X, Y, Z, respectively. Let the circumcircle of $\triangle ABC$ have center O and let the circumcircle of $\triangle XYZ$ have center I. Then let these two circles intersect at M, N, and let BC intersect MN at J. Then draw the two tangents from J to the two circumcircles that only intersect one circle. Let the point of tangency to the circumcircle of $\triangle ABC$ be K and let the point of tangency to the circumcircle of $\triangle XYZ$ be L. Then let the circumcenter of $\triangle JKL$ be H, and let the circumcircle of $\triangle JKL$ intersect the circumcircles of $\triangle ABC, \triangle XYZ$ at E, F, respectively. Then let the midpoint of EK be Q and the midpoint of FL be R. Prove H, O, Q and H, I, R are collinear.

(Assume all mentioned points exist.)

Number Theory

1 (Main) Prove that $31 | 5^{31} + 5^{17} + 1$.

1.1 (Extension) Prove that $x^2 + x + 1 | x^{31} + x^{17} + 1$.

1.2 (Generalization) For natural numbers a, b, c, prove that $x^2 + x + 1 | x^{3a} + x^{3b+1} + x^{3c+2}$.

2 What is the largest integer value of n such that $1.01^2 - \frac{n^2}{10000}$ is greater than or equal to 1? (Chapter N Section T)

3 Find the remainder of $(1^3)(1^3 + 2^3)(1^3 + 2^3 + 3^3)...(1^3 + 2^3 + ... + 99^3)$ when divided by 101. (Chapter N Section T)

4 Have $\frac{1}{a} + \frac{1}{b} = \frac{2}{5}$ for integers a, b. Find all values of a that have a corresponding value of b that satisfies this equality.

5 Prove that $\gcd(2n+8, 3n-2)$ is never equal to 8.

6 How many integer values of $1 \leq x \leq 100$ makes $x^2 + 8x + 15$ divisible by 10?