MPP - 5/17/18

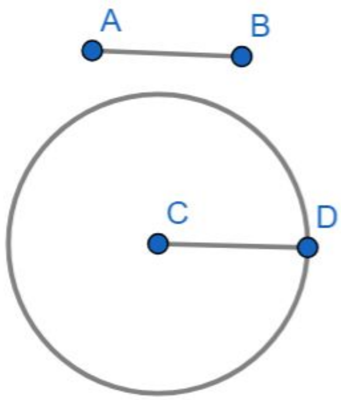
Welcome to the last session of the school year for MPP. We will be covering a wide range of topics to expand your mathematical and problem solving skills.

First we cover some basic constructions and the construction of a triangle given line segments of certain lengths. Let us first define construction. You are only allowed to use a compass and a straightedge (the former makes circles, and the latter makes straight lines). Points, lines, and circles can be made; you may make a line segment of certain lengths that are given on the paper by using your compass and spreading it out.

Construction 1 - Copy of a line segment

Given a line segment with length $x$, create another segment with length $x$.

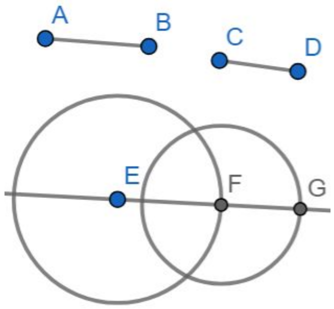
We reveal the answer as the following:




Take the compass and spread it out such that the length is the same as the given line segment. Drawing an arc and picking any other point gives it to us as desired. The reason this should be obvious enough.

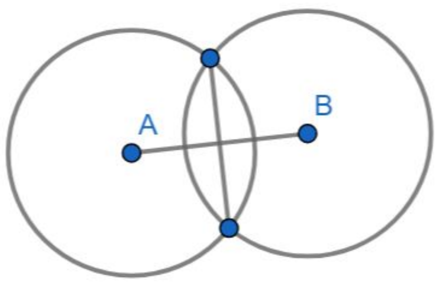
Construction 2 - Line segment with combined length of other line segments

We reveal the answer as the following:




Construction 3 - Perpendicular bisector of a line segment
Given a line segment, construct its perpendicular bisector.

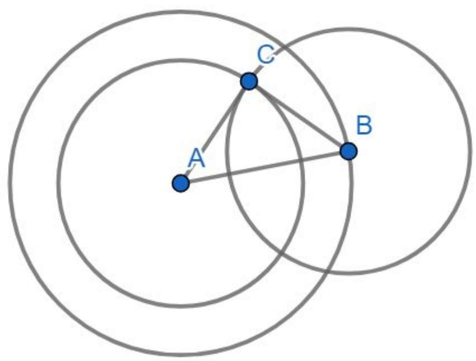
We reveal the answer as the following:
Draw two circles with radius greater than half the length of line segment. Then draw a line where the two circles intersect each other.




The reasoning is a bit lengthy, but in essence, this works because of isosceles triangles. (Note that the perpendicular bisector of a line is the locus of points equidistant from the endpoints.)

Construction 4 - Triangle given line segments
Given 3 line segments, construct a triangle such that each of its side lengths corresponds to the length of a line segment.

We reveal the answer as the following:

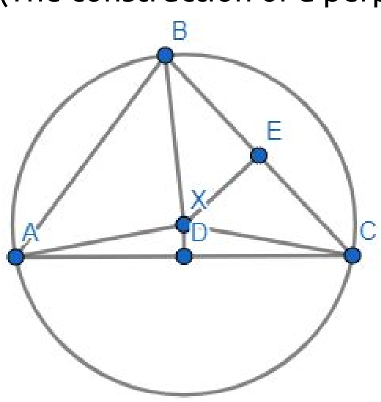



Have our three line segments have lengths $x, y, z,$ such that $x > y, z.$ Take any arbitrary point $A$ and draw a circle with center $A$ and length $x$ and draw another circle with center $A$ and length $y$ or $z.$ Then take an arbitrary point $B$ on the circle with radius $x$ and draw another circle with length $y$ or $z,$ this value being the one that was not used last time. They will intersect at $C,$ and connecting the lines gives us our desired triangle.

This works because we draw line segments with lengths $x, y, z.$

Construction 5 - Circumcircle of a triangle
Given a triangle, construct its circumcircle.

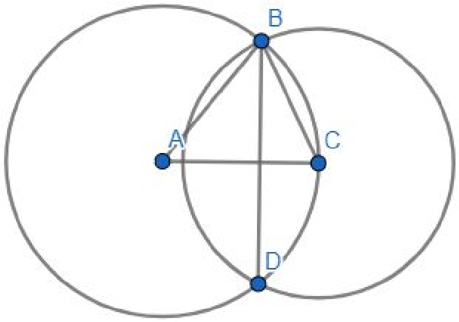
We reveal the answer as the following:
Construct two perpendicular bisectors of the sides of the triangle. They intersect at a point; draw a circle such that the center is on the point of intersection and a vertice. (The construction of a perpendicular bisector is Construction 3.)




This works because the circumcenter must be equidistant from all three vertices.

Construction 6 - Altitude of a triangle
Given a triangle, construct an altitude.

We reveal the answer as the following:
Have our triangle be $\triangle ABC$. Draw a circle centered at $A$ with radius $AB$ and a circle centered at $C$ with radius $CB$. They intersect at two points; the line formed by these two points is the altitude of $\triangle ABC$ respective to $B$.




This works because this reflects point $B$ across $AC$, and a line between a point and its reflection over a line is perpendicular to said line.

We shall now prove two theorems in Number Theory, being Fermat's Little Theorem and Wilson's Theorem. There are two proofs for Fermat's, both of which we shall introduce.

Fermat's Little Theorem

Given prime $p$, $a^p \equiv a \ (mod\ p)$, and if $gcf(a, p) = 1$, $a^{p-1} \equiv 1 \ (mod\ p)$.

Proof 1: Induction

Note that $1^p \equiv 1 \ (mod\ p)$. If $a^p \equiv a \ (mod\ p)$, then $(a+1)^p \equiv a^p + \binom{p}{1}a^{p-1} + \ldots + \binom{p}{p-1}a + 1 \equiv a + 1 \ (mod\ p)$. By induction, we are done.

Proof 2: Rearrangement

Note that $a, 2a, 3a \ldots (p-1)a$ is a rearrangement of $1, 2, 3 \ldots p-1$ when taken mod $p$. This is because if $ka \equiv ja \ (mod\ p)$, then $k \equiv j \ (mod\ p)$, yet there are no distinct terms that satisfy this. Then note this implies $(p-1)! \equiv a^{p-1}(p-1)! \ (mod\ p)$, and dividing by $(p-1)!$ yields $a^{p-1} \equiv 1 \ (mod\ p)$, as desired.

We will now prove Wilson's Theorem.

Wilson's Theorem

Given prime $p$, $(p-1)! \equiv -1 \ (mod\ p)$.

Proof:

Note that each $k$ in the list of numbers has an inverse. Note then that the inverse of every $k's$ inverse is $k.$ This means that for every pair of inverses, the product is $1.$ Note that there are $p-1$ terms in $(p-1)!$ However, one of these terms is $1$ which we can ignore, since it is its own inverse. So this gives us $p-2$ terms; they pair evenly, except for $p-1,$ which is its own inverse. So we have $(p-1)! \equiv p-1 \ (mod\ p),$ as desired.

Let's now discuss some combinatorics. We'll first talk about the container-item chart. There will be $c$ containers and $i$ items, and the amount of combinations in various cases will be shown.

| | Distinguishable Items | Indistinguishable Items |
|---|---|---|
| Distinguishable Bins | $c^i$ | $\binom{i+c-1}{c-1}$ (Stars and Bars) |
| Indistinguishable Bins | (Recursive function) $f(c,i) =$ $f(c-1, i-1) + i \cdot B(c-1, i)$ | $\frac{1}{(i+c)!} \sum_{n=0}^{c} (-1)^{c-n} \binom{c}{n} n^{(i+c)}$ |

Next we discuss derangement. Rather than being crazy, or mad, or something - hell, I don't even remember exactly what derangement means in popular culture - derangement, in the context of mathematics, means permuting (switching) objects around such that they don't appear in their original position (they change place).

Let's do some derangement problems.

Derangement 1: Teacher's Tests

Suppose Mrs. Crapapple has four tests from students $A, B, C, D.$ Since she is illiterate, she passes the tests out to students $A, B, C, D$ randomly. What is the probability **exactly** two students receive the same score?

Note that the amount of ways to permute the order of the tests is $4! = 24.$ Then note that the only way to get two incorrect tests given is to switch two tests up. There are $\binom{4}{2} = 6$ ways to do this, and since any of the $24$ permutations are equally likely, our answer is $\frac{6}{24} = \frac{1}{4}.$

Generalize: Teacher's Tests

Mrs. Crapapple has $x$ tests from $x$ students. She randomly passes them out. What is the probability $2$ students receive the same score?

This is left as an exercise to the student.

Derangement 2: The Airplane Problem

(This problem is probably a classical problem.)

There are 100 people boarding an airplane. The first person randomly sits in a seat. Everyone else boards their seat if it is available, and sits in a random empty seat if not. What is the probability the last person sits in his seat?

Let's think about what this problem is asking. How many different seats can be empty (the seat the last person is forced to be in)?

Lemma 1: Well, if he is in a seat $2-99,$ that means it was empty, but the person in said seat should have sat there. So the only seats available for the last person to sit in would be $1$ or $100.$

Lemma 2: Also, if someone's seat is taken who is not the first or last person, then both the first and last seat must be empty. Otherwise, there is a non-zero probability that both $1$ and $100$ are taken when the last guy sits down.

Since this is just some person $2-99$ choosing to sit in the first or last seat (or stalling), the probability is $\frac{1}{2}.$ (There is also an additional case where the first person chooses his seat or the last person's seat, but the bijection holds.)

Generalize: The Airplane Problem

There are $x \geq 2$ people boarding an airplane. The first person randomly sits in a seat. Everyone else boards their seat if it is available, and sits in a random empty seat if not. What is the probability the last person sits in his seat?

This is also left as an exercise to the student.

Now, we shall talk about telescoping. Mathematically, telescoping is basically cancelling stuff out and leaving behind a smaller number of terms. Consider the following question.

Telescoping 1: A Triviality

What is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot ... \cdot \frac{99}{100}$?

This is, as the title suggests, a triviality, so I won't bother to include the solution. Telescoping doesn't have to be hard. It can be trivial at times. Here are some harder telescoping problems; the first one was just to make telescoping seem less intimidating.

Telescoping 2: Triangular Reciprocals

Let $t_n$ depict the $nth$ triangular number. Find $\sum_{n=1}^{\infty} \frac{1}{t_n}$.

We want to find a way to cancel everything out. We're looking for a way to have everything cancel out. Since triangular numbers have a particular pattern, perhaps differences of fractions would work out. Note that $t_n = \frac{2}{n(n+1)}$. Then note that $\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$. Applying this gives us $\sum_{n=1}^{\infty} \frac{1}{t_n} = (\frac{2}{1} - \frac{2}{1+1}) + (\frac{2}{2} - \frac{2}{2+1}) + ... = \lim_{x \to \infty} \frac{2}{1} - \frac{2}{x+1} = \frac{2}{1} = 2,$ which is our answer.

Telescoping 3: Tangents

Find $\prod_{x=0}^{359} \tan(x)$. (The $\tan$ function is in degrees.)

Note that $\tan(x) = \frac{1}{\tan(90-x)}$. This implies that

$$\prod_{x=0}^{359} \tan(x) = \prod_{a=0}^{44} \tan(a) \cdot \prod_{b=0}^{44} \frac{1}{\tan(b)} \cdot \prod_{c=91}^{134} \tan(c) \cdot \prod_{d=91}^{134} \frac{1}{\tan(d)} ... \cdot \tan(45) \cdot \tan(90) \cdot \tan(135) \cdot ... \cdot \tan(360).$$

Now note that all of these products cancel out due to the definition of reciprocals and the application of $\tan(x) = \frac{1}{\tan(90-x)}$ on the multiples of $45$, and we are left with

$$\prod_{x=0}^{359} \tan(x) = 1.$$

Telescoping 4: Sines

Find $\sum_{x=0}^{359} \sin(x)$. (The $\sin$ function is in degrees.)

This is left as an exercise to the student; this can be done the same way as Telescoping 3.

Well, this is it. The end of MPP. I hope you've enjoyed coming to this class and I hope you come next year; it will remain free, so I see no reason why not!