Telescoping Handout

Preface

Given the generally positive reception of my Algebra Inequalities handout, I have decided to make a telescoping handout. While this packet is also written for a wide range of people, the difficulty of the problems will be narrower, as this handout will only cover one topic. Nonetheless, there will still be a handful of problems that are easier, and a handful of problems that are harder. Do not let this discourage you; exhaust all methods that you can think of before skipping a problem or looking at the solutions.

The text is formatting very similarly to my previous handout. There is a Theory component, Examples, and Exercises. The Theory is, in a proverbial sense, meant to give you the tools to telescope, while the Examples are meant to teach you how to use those tools, and the Exercises are meant to give you an opportunity to use these tools.

I hope this handout is of use to you. The solutions are available here.

Theory

Telescoping is the art of cancelling terms out in a long chain of operations to produce a shorter expression, usually only with part of the starting and ending terms. As telescoping is more of a technique, there is no rigorous definition for it. Perhaps the most famous example of telescoping are problems in the form $\frac{a}{a+1} \cdot \frac{a+1}{a+2} \cdot \ldots \cdot \frac{a+n-1}{a+n}$. Cross cancelling yields us with $\frac{a}{a+n}$. However, not all telescoping problems are so trivial; take, for example, problems in similar form to $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{6 \cdot 7}$. The following can be expressed as $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{6} - \frac{1}{7})$, and cancelling yields $\frac{6}{7}$. Usually, the bulk of a telescoping problem lies in figuring out how to express an expression in a telescoping form; once that is done, the rest is usually trivial.

Examples

1. Find $\frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \ldots \cdot \frac{10}{12}$.

Solution: Cancelling like terms, the following telescopes to $\frac{3 \cdot 4}{11 \cdot 12} = \frac{1}{11}$.

2. What is $\sum_{n=1}^{21} \frac{1}{n(n+2)}$, to the nearest integer?

Solution: Note that $\frac{1}{n(n+2)} = \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$. We see that the expression becomes $\sum_{n=1}^{21} \frac{1}{2}(\frac{1}{n}) - \sum_{n=3}^{23} \frac{1}{2}(\frac{1}{n})$, which telescopes to $\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{24} - \frac{1}{2} \cdot \frac{1}{25}$. Since $\frac{1}{48}$ and $\frac{1}{50}$ are sufficiently small, the following rounds to 1.

3. Find $\sum_{n=1}^{13} \frac{1}{n(n+3)}$.

Solution: We see that $\frac{1}{n(n+3)} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n-3})$. Applying this expansion, the expression telescopes to $\frac{1}{3}(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{14} - \frac{1}{15} - \frac{1}{16}) = \frac{2743}{5040}$.

4. Find $\frac{1}{2^2-1} + \frac{1}{3^2-1} + ... + \frac{1}{1000^2-1}$.

Solution: Rewriting, we see the following is $\frac{1}{(2-1)(2+1)} + \frac{1}{(3-1)(3+1)} + ... + \frac{1}{(1000-1)(1000+1)}$. Expanding using partial fractions gives us the following as $\frac{1}{2}([\frac{1}{1} - \frac{1}{3}] + [\frac{1}{2} - \frac{1}{4}] + [\frac{1}{3} - \frac{1}{5}] + ... + [\frac{1}{999} - \frac{1}{1001}])$. Telescoping then gives us our sum of $\frac{1}{2}(\frac{1}{1} + \frac{1}{2} - \frac{1}{1000} - \frac{1}{1001}) = \frac{1499499}{2002000}$.

5. What is $\frac{1}{\sqrt{4}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{10}} + ... + \frac{1}{\sqrt{397}+\sqrt{400}}$?

Solution: Multiply the numerator and denominator of the fraction of form $\frac{1}{\sqrt{n}+\sqrt{n+3}}$ by $\sqrt{n+3} - \sqrt{n}$. This gives us $\frac{\sqrt{n+3}-\sqrt{n}}{3}$. We may now telescope the sum to get $\frac{\sqrt{400}-\sqrt{4}}{3} = 6$.

6. Consider a square of side length 10. A square with side length 8 is taken out of it. A square with side length 6 is put back into it. A square with side length 4 is taken out of it. A square with side length 2 is put back into it. Find the total area of the resulting figure.

Solution: The following becomes $10^2 - 8^2 + 6^2 - 4^2 + 2^2 - 0^2$. We may factor out to get $(10+8)(10-8) + (6+4)(6-4) + (2+0)(2-0) = (10+8)2 + (6+4)2 + (2+0)2$. Distributing, this becomes $2(2+4+6+8+10) = 4(1+2+3+4+5) = 4 \cdot \frac{5(5+1)}{2} = 60$.

7. Have $f(100) = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - ... + \frac{99}{100}$ and $g(100) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ... + \frac{1}{100}$. Find $f(100) + g(100)$.

Solution: Note that adding fractions with the same denominator yields $(1-1)+(1-1)+...+(1-1)+1.$ This telescopes to $1,$ which is our answer.

Exercises

1. Find $\frac{1}{4}+\frac{1}{10}+\frac{1}{18}+\frac{1}{28}+\frac{1}{40}+\frac{1}{54}+\frac{1}{70}+\frac{1}{88}+\frac{1}{108}.$

2. Simplify $\sum_{n=a}^{b}\frac{1}{n(n+k)}$ in terms of $a, b, k.$

3. Find the sum of the reciprocals of the first 13 triangular numbers.

4. Simplify $\sum_{n=2}^{25}\frac{-3}{n(n+1)}.$

5. Find $\frac{1}{1\cdot2}+\frac{2}{2\cdot4}+\frac{3}{4\cdot7}+\frac{4}{7\cdot11}+\frac{5}{11\cdot16}.$

6. Find $1000^2-995^2+990^2...+10^2-5^2.$

7. Have $f(n)=\sum_{i=2}^{n}\frac{i-1}{i}\cdot(-1)^{(i-1)}$ and $g(n)=\sum_{i=2}^{n}\frac{1}{i}\cdot(-1)^{(i-1)}.$ Find the value of $\frac{1}{f(3)}+\frac{1}{g(3)}+\frac{1}{f(5)}+\frac{1}{g(5)}+...+\frac{1}{f(99)}+\frac{1}{g(99)}.$

8. Using the definition of $f(n)$ and $g(n)$ in the previous problem, find $\sum_{n=2}^{100}f(n)+g(n).$

Hints

1. Express the following as $\frac{1}{1\cdot4}+\frac{1}{2\cdot5}+...+\frac{1}{9\cdot12}.$ How can each individual term be expressed as the difference of two fractions?

2. Express $\frac{1}{n(n+2)}$ as a difference of two fractions. Then, telescope and use inequalities to round.

3. Think about the formula for a triangular number. Then, take its reciprocal and express as a difference of two fractions. Applying this gives us a telescopable sum.

4. Factor out the -3 to get $-3(\sum_{n=2}^{25} \frac{1}{n(n+1)})$. Then note that this is equivalent to $-3(\sum_{n=2}^{25} \frac{1}{n} - \frac{1}{n+1})$. What does this telescope to?

5. The larger distance between some of the factors in the denominator will make the value be a fraction of a difference of fractions. However, the numerator offsets this. What should be done with a chain of differences of numbers?

6. Try to express each difference of squares as a product of 5 and another sum of numbers. Doing this gives us $(1000-995)(1000+995)+...+(5-0)(5+0)$. What does this become? Can you find the value of $0+5+10+...+1000$?

7. Note $\frac{1}{f(n)} + \frac{1}{g(n)} = \frac{f(n)+g(n)}{f(n)g(n)}$. What is $f(n)+g(n)$ for all odd n?

8. We know what $f(n)+g(n)$ is for odd n from Exercise 7. What is $f(n)+g(n)$ for even n? Apply this to get the answer.