IGP Full

Dennis Chen

9-18-2018

0 Warmups

1. Consider rectangle $ABCD$ with $AB = 6, BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM, BN intersect AC at X, Y. Find XY.

2. Consider $\triangle ABC$ with $AB = 13, BC = 15, CA = 14$. If M is the midpoint of BC and P is a point on AC such that $MP \perp AC$, find MP.

   Variation: Consider $\triangle ABC$ with $AB = 13, BC = 15, CA = 14$. If M is the midpoint of AB and P is a point on AC such that $MP \perp AC$, find MP.

1 Area of a Triangle

1. Prove that
$$[ABC] = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

2. If two side lengths of a triangle are given to be 10 and 11, what is the maximum possible area of this triangle?

3. Prove $[ABC] = \frac{abc}{4R}$.

4. Prove $[ABC] = \frac{1}{2}ab \sin C$.

5. Prove $[ABC] = rs$.

6. A triangle has side lengths 4 and 8, and it has an area of $3\sqrt{15}$. Find the possible lengths of the third side.

7. Find the length of the altitude to the 14 inch side of a triangle whose two other sides have lengths of 13 inches and 15 inches.

8. Tangents from point C to circle O are extended to A and B such that AB is tangent to O at X. If the perimeter of $\triangle ABC$ is 50 and $[ABC] = 100$, find the area of circle O.

2 Triangle Centers

1. In trapezoid $ABCD$ with $BC||AD$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37°, \angle D = 53°$, and M, N be the midpoints of BC and AD respectively. Find the length MN.
2. Consider $\triangle ABC$ with $AB = 5$, $BC = 12$, and $AC = 13$. Angle bisector AD and median AE is drawn such that B, C, D, E are collinear. Find $[ADE]$.
3. The sides of $\triangle BAC$ are in the ratio $2 : 3 : 4$. BD is the angle bisector drawn to the shortest side AC, dividing it into segments AD and CD. If the length of AC is 10, then find the length of the longer segment of AC.
4. If triangle PQR has sides 40, 60, and 80, then the shortest altitude is K times the longest altitude. Find the value of K.

3 Telescoping

1. Find $\frac{1}{1\cdot 2} + \frac{2}{2\cdot 4} + \frac{3}{4\cdot 7} + \frac{4}{7\cdot 11} + \frac{5}{11\cdot 16}$.
2. Find
$$\frac{1}{1\cdot(1+2)} + \frac{1}{2\cdot(2+2)} + \cdots + \frac{1}{21\cdot(21+2)}$$
rounded to the nearest integer.
3. Simplify $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$.
4. Find $\frac{1}{4} + \frac{1}{10} + \frac{1}{18} + \frac{1}{28} + \frac{1}{40} + \frac{1}{54} + \frac{1}{70} + \frac{1}{88} + \frac{1}{108}$.
5. Find $\sum_{n=1}^{13} \frac{1}{t(n)}$, where $t(n) = \sum_{i=1}^{n} i$ ($t(n)$ is the nth triangular number).
6. If $f(x) = \frac{x^2}{x^2-1}$, find $\prod_{n=3}^{50} f(n)$.

4 Counting

1. A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?
2. Find the probability the product of the bottom face of 3 dice is composite.
3. How many 3 digit numbers have digits that when multiplied out, have an even product?
4. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

5. Brian shoots 6 balls, each making with a probability of p. Find the value of p that maximizes the probability that he makes exactly 3 of 6 shots.

6. How many 4 digit falling numbers are there? (A falling number is a number whose last digit is strictly smaller than its second-to last digit, and so on. Ex. 4321)