Definition

We define a **complex number** to be of the form $a + bi$ where $i = \sqrt{-1}$.

We define the **magnitude** of $z = a + bi$ as $|z| = \sqrt{a^2 + b^2}$.

We define $\Re(z) = a$ and $\Im(z) = b$ for $z = a + bi$. Note that $\Im(z)$ only considers the coefficient of the imaginary part!

We define the **complex conjugate** of z as $\overline{z}$ such that $z\overline{z} = |z|^2$. This implies if $z = a + bi$ then $\overline{z} = a - bi$.

We define the **polar form** of $z = a + bi$ as $(|z|, \theta)$ where $z = |z|(\cos\theta + i\sin\theta)$.

We define the **argument** of $z = (r, \theta)$ as θ.

We define **radians** as a measure of an angle such that π radians is equivalent to 180°.

We define the **roots of unity** as the n numbers $\omega_1, \omega_2 \cdots \omega_{n-1}, 1$ such that $\omega^n = 1$. Additionally, $\omega_1 = (1, \frac{2\pi}{n}), \omega_2 = (1, 2\frac{2\pi}{n})$, and so on.

We define **hyperbolic** cosine as $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Thus $\tanh = \frac{\sinh}{\cosh}$.

Formulas

For $z = (r, \theta)$ in polar form, $\Re(z) = r\cos\theta$ and $\Im(z) = r\sin\theta$.

The Triangle Inequality states that $|z| + |w| \geq |z + w|$, with equality if and only if their argument is the same.

For $z_1 = (r_1, \theta_1), z_2 = (r_2, \theta_2)$, we have $z_1 z_2 = (r_1 r_2, \theta_1 \theta_2)$. Magnitudes multiply and angles add.

De Moivre's Theorem states that $(r, \theta)^n = (r^n, n\theta)$. This is true for rational n.

Euler's Identity states $e^{ix} = \cos x + i\sin x$.

This implies that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $i\sin x = \frac{e^{ix} - e^{-ix}}{2}$.

We have $\cosh^2 x - \sinh^2 x = 1, \cosh 2x = \cosh^2 x + \sinh^2 x, \sinh 2x = 2\sinh x \cosh x.$

Techniques

Coefficient matching is very important! If we are given $a + bi = c + di,$ we are given two pieces of information: $a = c$ and $b = d.$

The roots of unity filter just relies on the fact that $\omega + \omega^2 + ... + \omega^{n-1} + 1 = 0$ for the nth roots of unity. This can be used to find roots of polynomials or factor polynomials, and to find certain combinatorial sums.