Preface

This packet was written for many different people and is intended to be as flexible as possible. As such, a wide range of difficulty is present in both the example problems and the exercises in the text. Do not let this discourage you; if you find a problem you cannot solve, read the solution and come back to it later. However, don't use the solutions as a way to avoid solving the problems -- only use it after you have exhausted all the methods you can think of, similar to how you would on an exam.

The text is organized by topic, which means that the difficulty of the text is not "top-down." Within each topic, there are three main sections: Theory, Examples, and Exercises. For formatting purposes, not every section has been labeled with the section they are in. For example, the Minimum/Maximum Problems section has examples mixed with the theory, which means it is not labeled. Generally, the Theory component will contain the key ideas that are needed to solve these problems, and information on terminology, whereas the Examples and Exercises contain problems that use the concepts discussed in the Theory component. Whereas the Theory is the concept the reader needs to know, the Examples section is how it is taught, and the Exercises are here to reinforce the reader's learning.

Happy reading! The solutions are available here.

QM-AM-GM-HM Inequality and Power Means

The QM-AM-GM-HM inequality states that for a set of positive reals $\{a_1, a_2 ... a_n\}$ that $\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}} \geq \frac{a_1+a_2+...+a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$, or, more formally, that $\sqrt{\sum_{i=1}^{n} \frac{a_i^2}{n}} \geq \sum_{i=1}^{n} \frac{a_i}{n} \geq \prod_{i=1}^{n} \sqrt[n]{a_i} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$, with equality occurring if and only if $a_1 = a_2 = ... = a_n$.

From left to right, the values in the inequality are known as the quadratic mean, arithmetic mean, geometric mean, and the harmonic mean of a set of positive reals. (The quadratic mean is also known as the root mean square, so you may see this inequality referred to as RMS-AM-GM-HM.)

An extension of QM-AM known as the Power Mean Inequality states that for a set of positive reals $\{a_1, a_2 ... a_n\}$ that $\sqrt[x]{\frac{a_1^x+a_2^x+...+a_n^x}{n}} \geq \sqrt[y]{\frac{a_1^y+a_2^y+...+a_n^y}{n}}$ if and only if $x \geq y$, with equality occuring at $x = y$.

Examples

1. Prove that $(a^2b + b^2c + c^2a) \geq 3abc,$ given that $\{a, b, c\}$ are all positive reals.

Solution: This is an example of a situation where we can directly apply AM-GM on the terms $\{a^2b, b^2c, c^2a\}$ and state that $\frac{a^2b+b^2c+c^2a}{3} \geq abc.$ The result follows after multiplying both sides of the inequality by 3.

2. Prove that $(a^5b + b^5c + c^5a)(ab^5 + bc^5 + ca^5) \geq 9a^4b^4c^4,$ given that $a, b, c > 0.$

Solution: Sometimes, you will have to apply an inequality on terms separately. Clever manipulation shows that applying AM-GM on $\{a^5b, b^5c, c^5a\}$ gives us $\frac{a^5b+b^5c+c^5a}{3} \geq a^2b^2c^2$ and applying AM-GM on $\{ab^5, bc^5, ca^5\}$ gives us $\frac{ab^5+bc^5+ca^5}{3} \geq a^2b^2c^2.$ Multiplying the two inequalities together gives us $\frac{(a^5b+b^5c+c^5a)(ab^5+bc^5+ca^5)}{9} \geq a^4b^4c^4.$ The result directly follows after multiplying both sides by 9.

3. Prove that $(a_1 + a_2 + ... + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}) \geq n^2$ given that $\{a_1, a_{2,} a_3 ... a_n\}$ are all positive reals.

Solution: We look for familiar terms in the QM-AM-GM-HM inequality. We notice that $(a_1 + a_2 + ... + a_n)$ is n times the arithmetic mean of $\{a_1, a_2 ... a_n\}$ and $(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n})$ is the denominator of the harmonic mean of $\{a_1, a_2 ... a_n\}.$ Seeing this, we notice that we want to use the AM-HM inequality to prove this. Dividing both sides by $n(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n})$ gives us $\frac{(a_1+a_2+...+a_n)}{n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}.$ The result directly follows from AM-HM.

4. What is the largest area of a rectangle with a perimeter of 20?

Solution: Have the sides be $a, b.$ This implies that $2(a + b) = 20,$ or $a + b = 10.$ By AM-GM, $\frac{a+b}{2} \geq \sqrt{ab}$ with equality occurring if and only if $a = b.$ Squaring, we see that $(\frac{a+b}{2})^2 \geq ab,$ and that ab is the value we want to maximize. Substituting in the value for $a + b = 10,$ we see that $25 \geq ab.$ Therefore, the maximum value of the rectangle is $25,$ where $a = b = 5.$

5. Find the minimum value of $3^x + 5^x + 2^x + 7.5^{-x} + 4^{-x}.$

Solution: Applying AM-GM gives us $\frac{3^x+5^x+2^x+7.5^{-x}+4^{-x}}{5} \geq \sqrt[5]{3^x 5^x 2^x 7.5^{-x} 4^{-x}} = 1$ with equality at $3^x = 5^x = 2^x = 7.5^{-x} = 4^{-x}$ which is achieved at $x = 0.$ Substituting yields an answer of $5.$

Exercises

1. Explain why QM-AM-GM-HM does not work when one or more of the values in the set $\{a_1, a_2 ... a_n\}$ is equivalent to 0.

2. Is QM-AM still true if one or more of the values in $\{a_1, a_2 ... a_n\}$ is equivalent to 0?

3. Prove that the area of a triangle with a fixed perimeter is maximized if the triangle is equilateral.

4. John is building a 8 mile long bridge, and wants to minimize the walking time. He can build a ramped section going up at a 45 degree angle, a ramped section going down at a 45 degree angle, or a flat section. The rate of walking for a section going up is 4 miles per hour, the rate of walking for a section going down is 2 miles per hour, and the rate of walking on a flat area is x miles per hour. Given that the altitude of the ending location of the bridge must be the same as the starting location's altitude, and that any bridge that can be constructed has the same amount of walking time, find x.

5. Positive reals $\{a_1, a_2 ... a_{100}\}$ multiply out to $\frac{13^{50}}{7^{50}}$. Find the smallest possible value of $a_1 + a_2 + ... + a_{100}$.

6. Prove that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{99}}{a_{100}} + \frac{a_{100}}{a_1} \geq 100$ given that $a_1, a_2 ... a_{100} > 0$.

Minimum-Maximum Problems

In the previous section, we introduced the idea of QM-AM-GM-HM. The inequality that will be most useful is probably AM-GM -- it is a versatile tool for problems such as maximization of area. Substituting sides for variables and multiplying them for area *generally* shows that to maximize the area, you want all of the side lengths to be equal.

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1. John wants to make a rectangle's boundary using 400 feet of wood. However, he has to run it through a processor that squishes the longer side into $\frac{1}{5}$ of its original length and the shorter side into $\frac{1}{2}$ of its original length. Find the maximum area of the rectangle he creates after he runs it through the processor.

Solution: Have the original dimensions be a, b with $a \geq b.$ Note that we are trying to maximize $\frac{a}{5} \cdot \frac{b}{2},$ which implies we are also trying to maximize $\sqrt{ab}.$ By AM-GM $\frac{a+b}{2} \geq \sqrt{ab}.$ Since $a+b$ is fixed, equality occurs at $a = b.$ This implies our original shape is a square of side length 100. Running it through the processor gives us a rectangle with dimensions 20 and 50. Multiplying, we see our final answer is $1000.$

However, problems such as maximum area are not all that fall under this type of problem. Another type of problem is minimum/maximum of a parabola. Take the following problems below.

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2. What is the maximum value of $-x^2+4x+45?$

Solution: The approach to solving this problem is similar to completing the square. Note that the following is equivalent to $-(x-2)^2+49.$ Since $(x-2)^2 \geq 0,$ $-(x-2)^2 \leq 0,$ and the maximum value $-(x-2)^2$ can take is 0. This means our maximum value is $0+49=49.$

3. What is the minimum value of $x^2+5x+8?$

Solution: As above, we complete the square and see that $x^2+5x+8=(x+\frac{5}{2})(x+\frac{5}{2})+\frac{7}{4}.$ Since a square is always non-negative, the minimum value for $(x+\frac{5}{2})^2$ is $0.$ Substituting, we see our minimum value is $\frac{7}{4}.$

4. Given that $a, b > 0,$ find the minimum value of $\frac{a}{b}+\frac{b}{a}+2.$

Solution: Rewriting yields $\frac{a^2+b^2+2ab}{ab}=\frac{(a+b)^2}{ab}.$ By AM-GM $\frac{a+b}{2} \geq \sqrt{ab},$ therefore, $\frac{(a+b)^2}{4} \geq ab.$ Rearranging yields $\frac{(a+b)^2}{ab} \geq 4$ with equality if and only if $a=b.$ If $a=b$ then LHS becomes $\frac{(2a)^2}{a^2} \geq 4$ and equality is possible, which means the minimum value of $\frac{a}{b}+\frac{b}{a}+2$ is $4.$

Most minimum-maximum problems involve cleverly rearranging the terms. The two above are a great example of this. In general, you want to express the value you are trying to maximize or minimize as a function of one unknown, whether that be a variable or another function itself. Keep that in mind as you solve the exercises below; they are meant to be solved creatively.

Exercises

1. Why would you not be able to solve the problem *"Find the minimum/maximum value of polynomial* $p(x)$ *for any real value of x"* if $p(x)$ has an odd degree?

2. Use the same technique you used for the previous problem to prove that you can find the minimum/maximum of $p(x)$ if $p(x)$ has an even degree.

3. If polynomial $p(x)$ has an even degree, when is there a minimum for $p(x)$? When is there a maximum for $p(x)$?

4. Jim has a rectangular prism with a surface area of $6.$ Find the maximum volume of his rectangular prism.

5. Jim got a new rectangular prism which has edge lengths that add up to 32 and a surface area of $8.$ Prove that his rectangular prism cannot have a volume of 4 or greater.

6. Prove that $x^4 + 3x^3 + 6x^2 + 4x + 1 \geq 0$ for all real $x.$

7. Find the minimum value of $3 + 3^x + 4 + 4^x + 2 + 2^{-x} + 6 + 6^{-x}.$

The Cauchy-Schwarz Inequality

In the following text, we will only be discussing the elementary form of Cauchy. There are two other forms, namely the complex form and the general form. The elementary form of Cauchy states that given a set of reals $\{a_1, a_2 ... a_n, b_1, b_2 ... b_n\}$ that $(a_1b_1 + a_1b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2),$ or more formally, that $(\sum_{i=0}^{n} a_ib_i)^2 \leq (\sum_{i=0}^{n} a_i^2)(\sum_{i=0}^{n} b_i^2),$ with equality occurring if and only if there is some constant k that $a_ik = b_i$ for all $i = 0, 1 ... n.$

Cauchy has a variety of uses, and there are many ways to integrate or hide Cauchy into a problem. One of the most common ways to do that is to provide an inequality in the form of $(x_1 + x_2 + ... + x_n)^2 \leq n(x_1^2 + x_2^2 + ... + x_n^2).$ To prove this, we can rewrite the equation as following: $(1x_1 + 1x_2 + ... + 1x_n)^2 \leq (1^2 + ... + 1^2)(x_1^2 + x_2^2 + ... + x_n^2),$ at which point we may directly apply Cauchy. Recognizing this form is not difficult, but there are certain factors such as coefficients that may lead to a problem not using this inequality directly, but it may still integrate an inequality of this form.

Examples

1. Prove that $(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a})^2 \le 5(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{d^2}+\frac{d^2}{e^2}+\frac{e^2}{a^2}),$ provided that $a,b,c,d,e>0.$

Solution: Rearrange to $(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a})^2 \le (1^2+1^2+1^2+1^2+1^2)(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{d^2}+\frac{d^2}{e^2}+\frac{e^2}{a^2}),$ apply Cauchy, and we are done.

2. Prove that $(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}) \le ([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2)$ for $a_1,a_2\ldots a_n>0.$

Solution: Multiplying both sides by $(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})$ gives us $(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})^2 \le (\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2).$ Applying AM-GM on $\{\frac{a_1}{a_2},\frac{a_2}{a_3}\ldots\frac{a_n}{a_1}\}$ gives us $\frac{(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})}{n} \ge 1.$ Multiplying both sides by n gives us $(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2) \ge n([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2)$ which implies $n([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2) \ge (\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})^2.$ Rearranging so that we can apply Cauchy, we see that $(1_1^2+1_2^2+\ldots+1_n^2)([\frac{a_1}{a_2}]^2+[\frac{a_2}{a_3}]^2+\ldots+[\frac{a_n}{a_1}]^2) \ge (\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})^2,$ which is true by Cauchy, and we are done. (The key step was multiplying both sides by $(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1})$ and applying AM-GM on that. The rest is rather simple compared to that.)

3. Prove that the equality condition for $(x_1y_1+x_2y_2)^2 \le (x_1^2+x_2^2)(y_1^2+y_2^2)$ is that there must be some constant k such that $kx_i=y_i$ for $i=1,2.$

Solution: Since we desire equality, $(x_1y_1+x_2y_2)^2=(x_1^2+x_2^2)(y_1^2+y_2^2).$ Multiplying out gives us $(x_1y_1)^2+2x_1x_2y_1y_2+(x_2y_2)^2=(x_1y_1)^2+(x_1y_2)^2+(x_2y_1)^2+(x_2y_2)^2$ which implies that $2x_1x_2y_1y_2=(x_1y_2)^2+(x_2y_1)^2.$ Dividing by $x_1x_2y_1y_2$ on both sides gives us $2=\frac{x_1y_2}{x_2y_1}+\frac{x_2y_1}{x_1y_2}.$ By AM-GM, $\frac{\frac{x_1y_2}{x_2y_1}+\frac{x_2y_1}{x_1y_2}}{2} \ge \sqrt{\frac{x_1y_2\cdot x_2y_1}{x_2y_1\cdot x_1y_2}}=1,$ with equality occurring if and only if $\frac{x_1y_2}{x_2y_1}=\frac{x_2y_1}{x_1y_2},$ implying $x_1y_2=x_2y_1.$ Rearranging gives us $\frac{x_1}{y_1}=\frac{x_2}{y_2},$ which means that there must be some constant k such that $kx_i=y_i$ for $i=1,2,$ which is what we desired.

4. Prove that a square of side length $a+b+c$ has an numerical area that is strictly less than the numerical surface area of a cube with side length $a,$ a cube with side length $b,$ and a cube with side length $c.$

Solution: We are trying to prove that $(a+b+c)^2 < 6(a^2+b^2+c^2)$. By Cauchy's Inequality, $(a+b+c)^2 \leq (1^2+1^2+1^2)(a^2+b^2+c^2)$. Multiplying RHS by 2 gives us a strict less than inequality, which is what we desired.

Exercises

1. Describe the sets of terms $\{a_1, a_2 ... a_n, b_1, b_2 ... b_n\}$ that can be 0 such that $(a_1b_1 + a_1b_2 + ... + a_nb_n)^2 = (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$ is true for all other terms in $\{a_1, a_2 ... a_n, b_1, b_2 ... b_n\}$ that are not equivalent to 0.

2. Prove that the product of the sum of the first n cubes and n is greater than or equal to the square of the sum of the first n triangular numbers.

3. Consider sets $A = \{a_1, a_2 ... a_n\}$ and $B = \{b_1, b_2 ... b_n\}$, such that $a_i = \pm 1$ and $b_i = \pm 1$ for integers i such that $1 \leq i \leq n$. How many distinct pairs of sets A and B exist such that $\{a_1, a_2 ... a_n\}$ can be mapped in any order to $\{x_1, x_2 ... x_n\}$ and $\{b_1, b_2 ... b_n\}$ can be mapped in any order to $\{y_1, y_2 ... y_n\}$ such that $(x_1y_1 + x_2y_2 + ... + x_ny_n)^2 = n(y_1^2 + y_2^2 + ... + y_n^2)$, in terms of n? (Remember that order does not matter in a set, all that matters is the contents.)

4. Verify that the equality condition for Cauchy is true.

Hints

The hints section is meant to assist you with solving the exercises if you have difficulty solving them, without giving the entire answer away. The reason these hints exist is to provide an intermediate step that guides you towards the solution, as it is better to solve some of a problem than none of it. Yet the hints should be used with just as much discretion if not more than the solutions; whereas the solutions contain the answer, the hints contain the motivation.

QM-AM-GM-HM Inequality and Power Means

1. Create a set containing a zero. What happens when a number is divided by zero? When it is multiplied by zero? Why does this take away the meaning of the QM-AM-GM-HM inequality?

2. How can you relate this to the QM and AM of another set of reals such that the AM of the original set is the same as the AM in the other set but the QM of the original set is greater than the QM of the other set?

3. Use Heron's formula to relate the side lengths of a triangle to its area. What should AM-GM be applied on? What is the value of $3s-a-b-c$?

4. What is her average speed after walking x miles up and x miles down? Remember that she walks for $\frac{1}{4}$ of the time on the way up. What connection does this have with the harmonic mean of 2 and 8?

5. Applying AM-GM on $\{a_1, a_2 ... a_{100}\}$ gives us $\frac{a_1+a_2+...+a_{100}}{100} \geq \sqrt[100]{a_1 a_2 ... a_{100}}$. How could we relate this to the value of $a_1 a_2 ... a_{100}$?

6. Applying AM-GM on $\{a_1, a_2 ... a_{100}\}$ directly gives us nothing useful. What could we apply AM-GM on instead to yield a useful result, such as an inequality where one side is a constant?

Minimum-Maximum Problems

1. What happens as $\lim_{x \to \infty}$? How about when $\lim_{x \to -\infty}$?

2. Depending on the coefficient, $\lim_{x \to \pm\infty} = \pm\infty$. When $x = \infty$ and $x = -\infty$ the end behavior is the same. What does this say about the smallest/largest value of y as $p(x)$ is a polynomial graph?

3. Again, think about the end behavior of the graph. When does it approach ∞? When does it approach $-\infty$?

4. Have the sides of his rectangular prism be a, b, c. What is the surface area of his rectangular prism, and what is its volume, in terms of a, b, c? Are there any expressions we can substitute in with constants?

5. Apply AM-GM. What is the only way for the prism to have an area of 4? Why is this condition never met?

6. How can we rearrange and factor this expression such that out inequality can be proved by casework? Which cases would we need? Remember that any real expression to an even power must be positive!

7. Look at AM-GM example 5. Note that if we want to minimize or maximize $f(x)+k$ for some variable $f(x)$ and some constant k we also want to minimize/maximize $f(x)+j$ for any constant $j,$ including $0.$

The Cauchy-Schwarz Inequality

1. What is the equality condition for Cauchy? How can it be achieved with a certain set of $0's,$ if it even can be?

2. Look for a connection between a triangular number and a summation of cubes. Then, a clever substitution and some manipulation will give us a form of Cauchy.

3. Rearrange this into Cauchy using the fact $n = x_1^2 + x_2^2 + ... + x_n^2$ under the conditions imposed on $x.$ Think about the equality condition for Cauchy. When will there be a common ratio? What choices of common ratios do we have? How does set B depend on set A? Make sure you do not overcount.

4. Plug in $a_i = b_i k$ for some arbitrary i and k into the equality. Clever manipulation should do the trick afterwards.